An Introduction to Point-Set Topology

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0. Introduction

Overview 0.1: This document will contain many of the definitions that are included in a standard introductory topology course. It will cover the typical types of topologies, continuous functions and metric spaces, compactness and connectedness, and the separation axioms. In addition to the definitions, I have included proofs to various exercises and theorems as well as some commentary explaining the intuition behind the definitions/theorems/propositions. I found the theorem-proof style of writing insufficient for building a strong understanding of the math, so I wanted to include my own explanations in addition to the theorems and definitions.

Prerequisites 0.2: The only prerequisites for these notes is elementary set theory that you learn in a discrete math class.

1. Topological Spaces

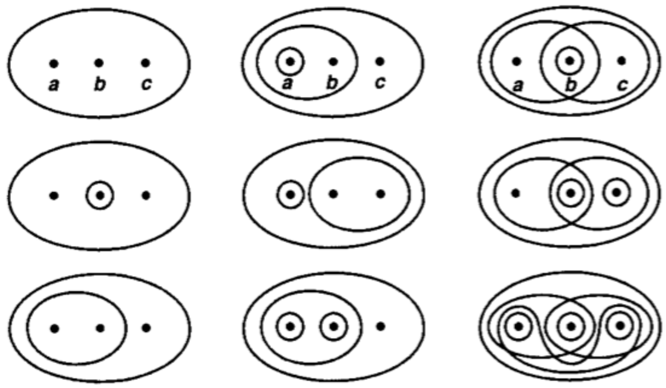
Definition 1.1: A *topology* on a set X is some collection $\mathcal{T}$ of subsets of X such that

(1) $\emptyset, X \in \mathcal{T}$
(2) The intersection of elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$
(3) The union of arbitrarily many elements of $\mathcal{T}$ is in $\mathcal{T}$

Intuitively, a topology is a structure that we impose on some set, much like the algebraic structure of numbers like integers and the reals. Just as numbers can seem meaningless without operations defined on them, sets without structure like a topology are, in a sense, vacuous.

Figure 1.2:

EXAMPLE 1. Let X be a three-element set, $X = \{a, b, c\}$. There are many possible topologies on X, some of which are indicated schematically in Figure 12.1. The diagram in the upper right-hand corner indicates the topology in which the open sets are X, $\varnothing$, $\{a, b\}$, $\{b\}$, and $\{b, c\}$. The topology in the upper left-hand corner contains only X and $\varnothing$, while the topology in the lower right-hand corner contains every subset of X. You can get other topologies on X by permuting a, b, and c.



Exercise 1.3: *Let X be a set, and let* $\mathcal{T} = \{U \mid X - U \text{ is countable or is all of } X\}$ *be some collection of subsets of X. Does this collection form a topology on X?*

Proof: If we let $U = X$, then $X - U = \emptyset$ and the empty set is countable, so it is in $\mathcal{T}$; let $U = \emptyset$, then $X - U = X$, and by definition of $\mathcal{T}$, $X \in \mathcal{T}$ as well, so we have verified property (1) of the definition of a topology. Now suppose $\{U_1, U_2, \ldots, U_n\}$ is some collection of sets $U_i \in \mathcal{T}$. Then, to show that the finite intersection of elements of $\mathcal{T}$ are in $\mathcal{T}$, we show that

$$\mathrm{X} - \cap_{i=1}^{n} (U_i) = \cup_{i=1}^{n} (X - U_i)$$

Note, any finite union of countable sets is also countable, so the latter set is countable, and thus in $\mathcal{T}$.
To show that arbitrary union of sets in $\mathcal{T}$ is in $\mathcal{T}$, we take an indexed family of sets $\{U_i\}, i \in I$ for some index family I. Then, we show that

$$X - \cup (U_i) = \cap (X - U_i)$$

Now, if $U_i \in \mathcal{T}$, then $X - U_i$ is countable. The arbitrary intersection of countable sets is countable, and thus the latter set is in $\mathcal{T}$, and we are finished.

Definition 1.4: The *discrete topology* on a set X is defined as $\mathcal{T} = \mathcal{P}(X)$, the power set of X. The *indiscrete topology* or *trivial topology* on a set X is defined as $\mathcal{T} = \{\emptyset, X\}$.

Definition 1.5: An *open set* A of some set X with topology $\mathcal{T}$, is defined precisely as a subset of X, as long as A is in $\mathcal{T}$. If A is not in $\mathcal{T}$, then A is not an open set of X. A set B of X is *closed* if $X - B \in \mathcal{T}$, i.e., its complement is open.

Note, this requires that both the whole set and the empty set be both open *and* closed in any arbitrary topology. It seems counterintuitive, but a set being open is not the negation of a set being closed (sometimes, you can even have a set that is *neither* open nor closed).

Exercise 1.6: *Let X be a topological space; let A be a subset of X. Suppose that for each* $x \in A$, *there is an open set U, such that* $x \in U, U \subset A$. *Show that A is open in X.*

Proof: The set A is composed of arbitrarily many points x_i. By hypothesis, for every x_i, there is an open set U_i such that $x_i \in U_i$, $U_i \subset A$. Then, $A = \cup\,(U_i)$, a union of open sets, and is thus open.

If you want to show something is open or closed, you must use some set theory to manipulate what you're given to show that it is in the topology (or its complement is). This previous example was quite simple, but the ones you might see in the future can be more involved. If you're ever uncertain, start from the basic definition of open, and work from there.

Definition 1.7: A *basis* for a topology on a set X is some collection $\mathfrak{B}$ of subsets of X such that

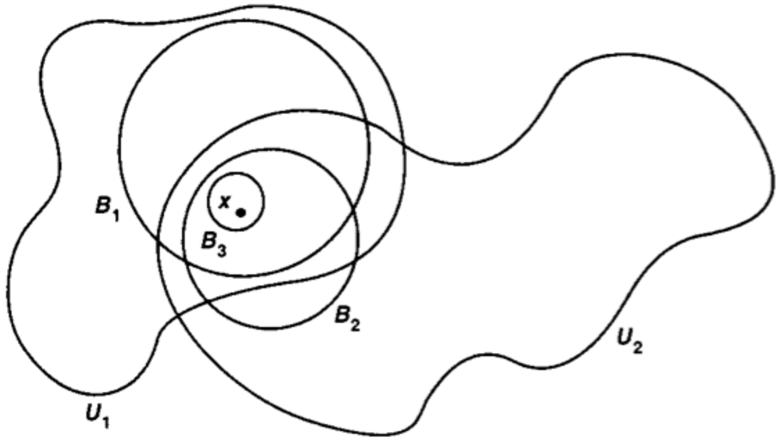
(1) For every $x \in X$, there is an element $B \in \mathfrak{B}$ such that $x \in B$ and
(2) If $x \in B_1$ and $x \in B_2$, then there is an element B_3 such that $x \in B_3 \subset B_1 \cap B_2$

The *topology generated by* $\mathfrak{B}$ is defined as: for every open set $U \subset X$ and $\forall x \in U$, there is a basis element $B \in \mathfrak{B}$, such that $x \in B \subset U$.

The topological definition of basis is, in a way, quite similar to the one used in linear algebra. Just as every element in some vector space can be written as a linear combination of basis vectors, every open set in some *topological* space can be written as a *union* of basis elements. It is analogous to being the spanning set, though not necessarily minimal, as in linear algebra. Bases are very valuable, as they can describe a topology with relatively little information itself (refer to Def. 1.14). The motivation behind the idea of a basis was to find a way to encode the structure of a set without enumerating (which is sometimes impossible!) all the elements of the topology itself.

Below is a diagram detailing the relationship of a basis (with elements B) relative to a topology (with elements U).

Figure 1.8:



Definition 1.9: Let X and Y be topological spaces. The *product topology* on $X \times Y$ has as a basis the collection $\mathfrak{B} = \{U \times V: U \text{ } open \text{ } in \text{ } X, V \text{ } open \text{ } in \text{ } Y\}$.

Theorem 1.10: *If $\mathfrak{B}$ is a basis for X, and $\mathcal{C}$ is a basis for Y, then $\mathfrak{D} = \{B \times C \mid B \in \mathfrak{B}, C \in \mathcal{C}\}$ is a basis for the product topology on $X \times Y$*

Proof: Suppose that $x \times y$ are some points in the open set $U \times V$ of $X \times Y$, $U \subset X, V \subset Y$. By definition of the product topology, U and V are open in X and Y, respectively. Then, by hypothesis, there is some $B \in \mathfrak{B}, C \in \mathcal{C}$ such that $x \in B \subset U, y \in C \subset Y$. Thus, there is some $B \times C$ such $x \times y \in B \times C \subset U \times V$. Therefore, $\mathfrak{D}$ is a basis for the product topology on $X \times Y$.

Definition 1.11: Let $\pi_1 : X \times Y \rightarrow X$ be the projection map defined as $\pi_1(x, y) = x$.

> Intuitively, what this map does is project some box in a "coordinate plane" onto one of the axes. You've probably dealt with similar maps in vector calculus or number theory, and this one behaves very similarly.

Definition 1.12: A map $f : X \rightarrow Y$ is an *open map* if for every open set $U \in X$, $f(U)$ is open in Y.

Proposition 1.13: *The projection map $\pi_1 : X \times Y \rightarrow X$ is an open map*

Proof: Recall, by Definition 1.9, an open set in $X \times Y$ with the product topology has a basis of the form $A \times B$, where A is open in X, and B is open in Y. Thus, for any open set $U \times V \in X \times Y$ $f(U \times V) = U$, which is, by definition, open in X.

Definition 1.14: The *standard topology* on $\mathbb{R}$ has, as a basis, the collection of all open intervals on the real line:

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

Typically, this is the topology that we use when discussing the real number line (hence the name), unless otherwise specified.

Up to this point, you might (or maybe not) have begun to wonder what happens to a topology on a space when we look at some subset of the space. That is, if we begin to "zoom in" on, or cut out a subset of a space, what happens to the topology? This natural question leads us to our next definition.

Definition 1.15: Let X be a topological space with topology $\mathcal{T}$, and let $Y \subset X$. Then, the collection

$$\mathcal{T}_{\mathrm{Y}} = \{\mathrm{U} \cap \mathrm{Y} \mid \mathrm{U} \text{ open in X}\}$$

forms what we call the *subspace topology* on Y. Additionally, when Y inherits this topology from X, we call it the *subspace* of X.

Theorem 1.16: *If A is a subspace of X, and B is a subspace of Y, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.*

Proof: Suppose A is a subspace of X and B is a subspace of Y. A and B have the topologies

$$\mathcal{T}_A = \{\mathrm{U} \cap A \mid \mathrm{U} \text{ open in X}\}$$

and

$$\mathcal{T}_B = \{V \cap B \mid \mathrm{V} \text{ open in Y}\}$$

respectively. Then the product topology on $A \times B$ has as a basis, by definition,

$$\mathfrak{B} = \{W \times Z \mid W\ open\ in\ A, Z \text{ open in } B\}$$

or equivalently,

$$\mathfrak{B} = \{(U \cap A) \times (V \cap B) \mid U \text{ open in } X, V \text{ open in } Y\}$$

Consider the basis for the topology $A \times B$ inherits as a subspace of $X \times Y$,

$$\mathcal{C} = \{(C \times D) \cap (A \times B) \mid C \times D\ open\ in\ X \times Y\}$$

But, since $C \times D$ is open in $X \times Y$ with the product topology, C is open in X and D is open in Y. Then, note that $(C \times D) \cap (A \times B) = (C \cap A) \times (D \cap B)$, so

$$\mathcal{C} = \{(C \cap A) \times (D \cap B) \mid C\ open\ in\ X, D\ open\ in\ Y\}$$

Thus, $\mathcal{C}$ is equivalent to $\mathfrak{B}$, and we are done.

Definition 1.17: Let A be a subset of a topological space X. The *closure* (denoted $\overline{A}$) of a set A is the intersection of all closed sets that contain A; or equivalently, it is the minimal closed set containing A. The *interior* (denoted $Int\,A$) of a set A is the union of all open sets contained in A.

The relationship between $Int\,A$, A, $\overline{A}$ is $Int\,A \subset A \subset \overline{A}$. If A is open, then $Int\,A = A$; if A is closed, then $A = \overline{A}$. You might be wondering "what happens if A is neither open nor closed?", as this definition doesn't make it clear what it means to be neither open nor closed. This idea will motivate an equivalent definition of closure.

Definition 1.18: Let A be a subset of a topological space X. A point $x \in A$ is a *limit point* of A, if every open set containing x intersects A in a point different from x (another term for an open set containing x is a *neighborhood of x*). The closure of a set A is $\overline{A} = A \cup A'$, where A' is the set containing all the limit points of A.

Suppose we have some circle A defined as

$$A = \{x, y \in \mathbb{R} \mid x^2 + y^2 < 1\}$$

The limit points of A are

$$A' = \{x, y \in \mathbb{R} \mid x^2 + y^2 \leq 1\}$$

Suppose we adjoin the point (0,1) to A; then $A \neq IntA$, and $A \neq \overline{A}$. We can intuitively understand the limit points of a set more easily now; in this case, if we take any open ball around the limit points of A, it will necessarily intersect the inside of the circle, thus verifying our definition for A'. This intuition won't always make sense, but it's strong when it does.

Note: Points in the interior can be limit points but need not be limit points. Try and find an example where some point in the interior of a set is not a limit point (what topology contains a set where a point is its own neighborhood?).

Lemma 1.19: Let *A* be a subset of some space *X*. Then, $x \in \overline{A}$ if and only if every open set *U* containing x intersects *A*.

Proposition 1.20: *Let A, B be subsets of some space X. If $A \subset B$, then $\overline{A} \subset \overline{B}$.*

Proof: Suppose $A \subset B$, and for the sake of contradiction $\overline{A}$ is not a subset of $\overline{B}$. Then $\exists x \in \overline{A}$ such that $x \notin \overline{B}$. If $x \in \overline{A}$, then every open set *U* containing x intersects *A*. But, by hypothesis, $A \subset B$, so if *U* intersects *A*, then *U* also intersects *B*. Thus, by Lemma 1.19, $x \in \overline{B}$; this contradicts our supposition, so no such *x* exists. Therefore, if $A \subset B$, then $\overline{A} \subset \overline{B}$.

Proposition 1.21: *Let A, B be subsets of some space X. Then,* $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof: Consider some arbitrary $x \in \overline{A \cup B}$. By Lemma 1.19, every open set U containing x intersects $A \cup B$, or equivalently, U intersects A or B. We know that for any set C, $C \subset \overline{C}$, so if U intersects $A \ or \ B$, U intersects $\overline{A} \ or \ \overline{B}$. Thus, by Lemma 1.19, $x \in \overline{\overline{A}} \ or \ x \in \overline{\overline{B}}$. Note, though, that $\overline{\overline{A}} = \overline{A} \ and \ \overline{\overline{B}} = \overline{B}$. So, if $x \in \overline{\overline{A}}$ or $x \in \overline{\overline{B}}$, $x \in \overline{A}$ or $x \in \overline{B}$. Therefore $x \in \overline{A} \cup \overline{B}$, so we may say $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. For the other direction, consider some arbitrary $x \in \overline{A} \cup \overline{B}$. Then, we say that $x \in \overline{A} \ or \ x \in \overline{B}$. Without loss of generality, assume $x \in \overline{A}$. Then, every open set U containing x intersects A; and thus, U intersects $A \cup B$. But then, by Lemma 1.19, $x \in \overline{A \cup B}$, so we know $\overline{A \cup B} \supset \overline{A} \cup \overline{B}$. Thus, by showing containment both ways, we know that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

2. Continuous Functions and Metric Spaces

Definition 2.1: Let X and Y be topological spaces. A function $f: X \to Y$ is said to be *continuous* if for every open subset $V \subset Y$, $f^{-1}(V)$ is open in X.

> This is a natural generalization of continuous functions in calculus/analysis. In calculus you are introduced to the epsilon-delta definition of continuity; for every n-dimensional ball of radius epsilon (no matter how small) around a point in the range, there is always a delta in the domain that maps to that ball. Under the standard topology on $\mathbb{R}$, this definition of continuity coincides with the epsilon-delta one we are all familiar with. However, we will see that this definition works in all topological spaces, even those without a metric. We will also see the epsilon-delta generalizes to all metric-spaces.

Proposition 2.2: *Let X and Y be topological spaces. A function $f: X \to Y$ is continuous if and only if for each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.*

Proof: Suppose $f: X \to Y$ is continuous, and for the sake of contradiction, there exists some neighborhood V of $f(x)$ such that there does not exist a neighborhood U of x where $f(U) \subset V$. Then, there does not exist some neighborhood W in X such that $W = f^{-1}(V)$. However, this contradicts the definition of continuity, so no such V exists. In the other direction, suppose that for each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$. Let V be some neighborhood of $f(x)$. Then, $x \in f^{-1}(V)$. Choose some neighborhood U_i such that $x \in U_i, f(U_i) \subset V$, which we can do by hypothesis. Then for all points $x \in f^{-1}(V)$, there exist corresponding open sets containing them. Take their union, and you have an open set

$$\cup \{U_i\}_{i \in I} = f^{-1}(V)$$

> This definition of continuity coincides a lot with our intuition of open balls that we see in the epsilon-delta definition of metric spaces. That is, for every

neighborhood (open ball) A of points in the range, there is a corresponding neighborhood (open ball) in the domain that maps into entirely A.

Exercise 2.3: *Prove that for functions* $f\colon \mathbb{R} \to \mathbb{R}$, *the epsilon-delta definition of continuity implies the open set definition.*

Proof: Consider some function $f\colon \mathbb{R} \to \mathbb{R}$; the epsilon-delta definition of continuity states that for any $\varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$. Now, note that this follows directly from Proposition 2.2, just with open intervals instead of neighborhoods. Rewritten in terms of open intervals, for some $\varepsilon > 0$, there is a neighborhood $U = (x_0 - \delta, x_0 + \delta)$ of x_0, such that $\forall x \in U$, $f(x) \in V$, where $V = (y_0 - \varepsilon, y_0 + \varepsilon)$ is some neighborhood of $f(x_0)$, and $f(x_0) = y_0$. With less notation, for some neighborhood *V* of any $f(x_0)$, there will always exist a neighborhood U of x_0, such that $f(U) \subset V$. By Proposition 2.2, this is equivalent to the open set definition and we are done.

Definition 2.4: Let *X* and *Y* be topological spaces. A function $f\colon X \to Y$ is a *homeomorphism* if f *and* f^{-1} are continuous and bijective. If *X* and *Y* have a homeomorphism between them, they are *homeomorphic*

> A homeomorphism is a structure preserving map, i.e., the topological version of an isomorphism. Notice, a homeomorphism requires f and f^{-1} to be continuous. Bijective functions that are continuous preserve a certain type of structure: the topology. If a set is open in *X*, it is mapped to an open set in *Y* and vice versa. This is analogous to, for example, a ring isomorphism, where you want $f(0) = 0$ (part of the structure of a ring are its identity elements, so this must also be preserved). Just like any isomorphism, topological spaces that are homeomorphic are, for all intents, the same spaces with different notation.

Proposition 2.5: *The subspace* (a, b) *of* $\mathbb{R}$ *is homeomorphic with* $(0,1)$.

Proof: For (a, b), let $f\colon (a, b) \to (0,1)$ be defined as $f(x) = \frac{x-a}{b-a}$. Now, to show f and f^{-1} are bijective and continuous. It suffices to show that f is bijective, as the bijectivity of f^{-1} will necessarily follow. To show surjectivity choose some $y \in (0,1)$. We must find an $x \in (a, b)$ such that $f(x) = y$. Define $x = y(b - a) + a$; this is a valid definition, as $0 < y < 1$, so we see $a < y(b - a) + a < b$, for any y. Thus, there will always be an $x \in (a, b)$ that satisfies $y(b - a) + a$. Then, $f(x) = f\big(y(b - a) + a\big) = \frac{y(b-a)+a-a}{(b-a)} = y$. Now for injectivity, choose some $f(x_1), f(x_2)$ such that $f(x_1) = f(x_2)$; we must show that $x_1 = x_2$. $f(x_1) = \frac{x_1-a}{(b-a)}$ and $f(x_2) = \frac{x_2-a}{(b-a)}$; therefore $\frac{x_1-a}{(b-a)} = \frac{x_2-a}{(b-a)}$, $x_1 - a = x_2 - a$, $x_1 = x_2$, which is what we desired. We know f and f^{-1} are bijective, and now must show that they're continuous.

Let's start with f. Choose some $x \in (a, b)$, and some neighborhood (c, d) of $f(x)$. Then, there exist some $y, z \in (a, b)$, such that $y < x < z$ and $f(y), f(z) \in (c, d)$ by definition of bijectivity and the function f. Then, let (y, z) be a neighborhood of x. It is obvious that $f\big((y, z)\big) \subset (c, d)$, so we can conclude that f is continuous. An identical argument follows for f^{-1}, so we are done. Therefore, the subspace (a,b) of $\mathbb{R}$ is homeomorphic to $(0,1)$.

Proposition 2.6: *Let $f: A \to B$ and $g: C \to D$ be continuous functions. Define $f \times g: A \times C \to B \times D$ as*

$$(f \times g)(a \times c) = f(a) \times g(c)$$

Then $f \times g$ is continuous.

Proof: The basis for the product topology of $X \times Y$ is $\mathfrak{B} = \{M \times N \mid M \textit{ open in } X, N \text{ open in } Y\}$. Take some set W open in $B \times D$; we need to show $(f \times g)^{-1}(W)$ is open in $A \times C$. Let W be defined as

$$\mathrm{W} = \cup_{(i,j)\in I\times J} \left(U_i \times V_j\right)$$

i.e., some arbitrary union of basis elements. Then,

$$\mathrm{W} = (\cup_{i\in I} U_i) \times \left(\cup_{j\in J} V_j\right) = \mathrm{U} \times V$$

where U_i is open in B, and V_j is open in D. Then,

$$(f \times g)^{-1}(W) = (f \times g)^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V)$$

But $f: A \to B$ and $g: C \to D$ are continuous, so $f^{-1}(U)$ and $g^{-1}(V)$ are open. Consequently, $(f \times g)^{-1}(W)$ is also open as it is a product of open sets; therefore $f \times g$ is continuous.

Definition 2.7: A *metric d* on some space is a function $d: X \times X \to \mathbb{R}$ having the properties for each $x, y, z \in X$

(1) $d(x,y) \geq 0, d(x,x) = 0$
(2) $d(x,y) = d(y,x)$
(3) $d(x,y) + d(y,z) \geq d(x,z)$

> A metric is a generalization of distance. Essentially, we chose the most important and general characteristics of distance to define a metric. Metric spaces are studied frequently in real analysis and won't be covered in great detail here. The principal question arising from this section is demonstrating that a metric induces a topology identical to ones previously encountered.

Definition 2.8: Let X be a metric space, and let d be the metric on X. The *ε-ball centered at x* is the set of all points such that, for some $\varepsilon > 0$, $d(x,y) < \varepsilon$. We denote this set

$$B_d(x,\varepsilon) = \{y \mid d(x,y) < \varepsilon\}$$

Definition 2.9: The *metric topology induced by d* is the topology with basis $B_d(x,\varepsilon) = \{y \mid d(x,y) < \varepsilon\}$, for all $x \in X, \varepsilon > 0$.

This definition of a metric topology looks similar to the standard topology on $\mathbb{R}$. In fact, we can see that, given what's called the standard metric on $\mathbb{R}$, $d(x, y) = |x - y|$, we can see that the definition of an open ball in $\mathbb{R}$, and an open interval in $\mathbb{R}$ are identical; that is, the metric topology induced by d and the standard topology are the same.

Exercise 2.10: The metric $d'(\boldsymbol{x}, \boldsymbol{y}) = |x_1 - y_1| + \dots + |x_n - y_n|$ induces the regular topology on $\mathbb{R}^n$

Proof: First, let's show that d' is a metric. The first two properties are fulfilled trivially. To show the triangle inequality, let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$. Then, we see that:

$$d'(\boldsymbol{x}, \boldsymbol{z}) = |x_1 - z_1| + \dots + |x_n - z_n|$$

and

$$d'(\boldsymbol{x}, \boldsymbol{y}) + d'(\boldsymbol{y}, \boldsymbol{z}) = |x_1 - y_1| + \dots + |x_n - y_n| + |y_1 - z_1| + \dots + |y_n - z_n|$$

or equivalently

$$d'(\boldsymbol{x}, \boldsymbol{y}) + d'(\boldsymbol{y}, \boldsymbol{z}) = |x_1 - y_1| + |y_1 - z_1| + \dots + |x_n - y_n| + |y_n - z_n|$$

But term by term, $|x_i - y_i| + |y_i - z_i| \geq |x_i - z_i|$, as in general, $|a - b| + |b - c| \geq |a - c|$. Therefore, as each term is in $d'(\boldsymbol{x}, \boldsymbol{y}) + d'(\boldsymbol{y}, \boldsymbol{z})$ greater than or equal to the corresponding term of $d'(\boldsymbol{x}, \boldsymbol{z})$, we may say that $d'(\boldsymbol{x}, \boldsymbol{y}) + d'(\boldsymbol{y}, \boldsymbol{z}) \geq d'(\boldsymbol{x}, \boldsymbol{z})$.

Now to show that d' induces the standard topology on $\mathbb{R}^n$. Consider some open set U of $\mathbb{R}^n$ and let $\boldsymbol{x} \in U$. We wish to find some $B_{d'}(\boldsymbol{x}, \varepsilon) \subset U$. Recall that U $= \prod_{i=1}^{n}(y_i, z_i)$ under the product topology of $\mathbb{R}^n$. Choose an ε, such that $\varepsilon = min\{\varepsilon_1, \dots, \varepsilon_n\}$, where $\varepsilon_i = min\{x_i - y_i, x_i - z_i\}$. Then, it is clear that $B_{d'}(\boldsymbol{x}, \varepsilon) \subset U$. For the other direction, I will only verbally explain proof. Consider some $B_{d'}(\boldsymbol{x}, \varepsilon)$; we wish to find an open set V such that $V \subset B_{d'}(\boldsymbol{x}, \varepsilon)$. For any $\varepsilon > 0$, there certainly exists a $\delta > 0$, such that $(x_1 - \delta, x_1 + \delta) \times \ldots \times (x_n - \delta, x_n + \delta)$ is "inside" the $B_{d'}(\boldsymbol{x}, \varepsilon)$. Constructing such a δ is less trivial though. Regardless, once you find such δ, the proof is complete, and d' induces the standard topology on $\mathbb{R}^n$.

3. Connectedness and Compactness

Definition 3.1: Let X be a topological space. A *separation of X* is some nonempty open sets $U, V \subset X$ such that $U \cap V = \emptyset$ and $U \cup V = X$. If a separation exists for a space X, then that space is said to be *disconnected*. If no such separation exists, then that space is *connected*.

The idea of connectedness is quite intuitive. It's easy to imagine some space X being composed of open sets, and if there is some partition of X, then the space can be divided cleanly, i.e., the word disconnected. However, if every open set is intertwined with another open set, like links in a chain, then we say that is connected.

Theorem 3.2: *The union of a collection of connected subspaces of X that share a point is connected.*

Proof: Suppose we have some family $\{A_n\}$ of connected subspaces and let x be a point in $\cap A_n$. Suppose for the sake of contradiction that $\cup A_n$ is not connected. Then, there are nonempty open sets C and D such that $C \cup D = \cup A_n \text{ and } C \cap D = \emptyset$. However, if C and D have an empty intersection, then either C or D do not contain any elements from $\{A_n\}$, as, by hypothesis, x is a point in $\cap A_n$ (if they both had an element from $\{A_n\}$, $x \in C \cap D$). But that cannot be the case, as C and D are nonempty. Thus, with the discovery of a contradiction, we can conclude that no such C and D exist, and $\cup A_n$ is connected.

Proposition 3.3: *Let $\{A_n\}$ be a sequence of connected subspaces such that $A_n \cap A_{n+1} \neq \emptyset$. Then $\cup \{A_n\}$ is connected*.

Proof: Note, this follows almost directly from Theorem 3.2, just in an inductive case. Consider $A_1 \text{ and } A_2$ as our base case. From Theorem 3.2, $A_1 \cup A_2$ is connected. For our inductive hypothesis, assume $A_1 \cup \ldots \cup A_n$ is connected. Then, as $A_n \cap A_{n+1} \neq \emptyset$, $(A_1 \cup \ldots \cup A_n) \cap A_{n+1} \neq \emptyset$. Thus, when we invoke our inductive hypothesis, there cannot exist a separation of $A_1 \cup \ldots \cup A_n \cup A_{n+1}$, so it is connected. Our induction is complete, and we know that for any sequence $\{A_n\}$ of connected subspaces such that $A_n \cap A_{n+1} \neq \emptyset$, $\cup \{A_n\}$ is connected.

Proposition 3.4: *A space is totally disconnected if the only connected subspaces are singletons. Then, some space X with the discrete topology is totally disconnected, but the converse is not true.*

Proof: Recall that a set X with the discrete topology $\mathcal{T}$ has as a topology the collection $\mathcal{P}(X)$. Suppose that A is an open set of X such that it contains at least 2 points. Select some one point $x \in A$, and denote all the other points in A as $\{y_i\} \in A$. By the definition of a power set, there exists a $B \in \mathcal{T}$ such that $\{x\} = B$ and $C \in \mathcal{T}$ such that $\{\{y_i\}\} = C$. Then, from the definition of subspace topology, $B \cap A = B, B \in \mathcal{T}_A$ and $C \cap A = C, C \in \mathcal{T}_A$. Therefore, there exist 2 open sets in A such that $B \cup C = A, B \cap C = \emptyset$. Consequently, any subspace of X with more than one element is disconnected. Thus, any space X with the discrete topology is totally disconnected. However, not every totally disconnected set has the discrete topology. At the end of this section, we will consider the Cantor set, which is totally disconnected, but does not have the discrete topology.

Proposition 3.5: *Let $Y \subset X$, and let X and Y be connected. If A and B form a separation of $X - Y$, then $Y \cup A, Y \cup B$ are connected.*

Proof: Suppose that X and Y are connected, A and B form a separation of $X - Y$, and, for the sake of contradiction, $Y \cup A$ or $Y \cup B$ is not connected. Without loss of generality, suppose $Y \cup A$ is not connected. Then, there exist sets U, V such that $U \cup V = Y \cup A,\ U \cap V = \emptyset$. Note, U and V are disjoint from B, as Y and A are disjoint from B. Define $W = U \cup B$. Then, we can see that $W \cup V = X$, (as $W \cup V = V \cup U \cup B = Y \cup A \cup B = X$). But W and V are a separation of X: V is disjoint from U and B, so V is also disjoint from their union. This contradicts our assumption that X is connected, and we thus know that $Y \cup A$ is connected. Because we proved this for a general set A, we also know that $Y \cup B$ is connected, and thus we are finished.

Definition 3.6: Let X be a topological space. A *covering* $\mathcal{C}$ of X is a collection of subsets of X such that $\cup\,(A \in \mathcal{C}) = X$. If $\mathcal{C}$ contains only open sets, then it is called an *open covering*.

Definition 3.7: Let X be a topological space. X is *compact* if for every open cover $\mathcal{C}$ of X, there is a finite subcover that also covers X.

> Compactness is less intuitive than connectedness. In a way, it conveys information about the "density" (dense means something else in topology but ignore that for now) of a space. Imagine you're a rich planter and you want to protect your crops from thieves. Obviously, you hire guards to protect your estate, but you can't hire too many guards or else you'll run out of money. So, you hire a finite number of guards to watch your crops. Now, your guards can only protect what they see. If, for some reason, some land is not watched, you will lose crops and ultimately run out of business. Note, though, you can have an infinite number of crops, they just have to be sufficiently "dense" such that a finite number of watchmen can protect them. To be compact is to have a set that is capable of being completely "watched" (covered) by a finite number of "watchmen" (open sets).

Proposition 3.8: *The finite union of compact subspaces of X is compact.*

Proof: Suppose that we have finite open subspaces of X, $A_1, \dots, A_n$. Then, for each cover of A_i, there exist finite subcovers $\{C_i\}_{i \in I}$ such that $A_i = \cup_{i \in I}\,(\mathcal{C}_i)$. Note, any covering of $A_1 \cup \dots \cup A_n$ must be the union of covers of each A_i. Note, though, every cover of each A_i has a finite subcover, so, for any finite union of any combination of covers, there exist a finite union of finite subcovers of $A_1 \cup \dots \cup A_n$, which is finite. Thus, $A_1 \cup \dots \cup A_n$ must be compact as every cover has a finite subcover (defined as the finite union of finite subcovers of A_i).

Definition 3.9: A set S in a metric space X is totally disconnected if for any distinct $x, y \in S$, there exist separated sets A and B such that $x \in A, y \in B, and\ A \cup B = S$.

Definition 3.10: Let X be a topological space. A point $x \in X$ is an *isolated point* if $\{x\}$ is open in X.

Theorem 3.11: *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Theorem 3.12: *Let X be a non-empty compact Hausdorff space. If X has no isolated points, then X is uncountable*

Definition 3.13: Let A_0 be the closed interval $[0,1]$ in $\mathbb{R}$. Define

$$A_n = A_{n-1} - \cup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n}\right)$$

Let C $= \cap_{n \in \mathrm{Z}_+} A_n$, which we call the Cantor set, or the Cantor ternary set.

Proposition 3.14: *The Cantor set is totally disconnected.*

Proof: Let $a, b \in C$ be distinct points in the Cantor set, and without loss of generality, let $a < b$. Then, consider the interval (a, b). By the construction of the Cantor set, we remove some nonzero length from this interval. Take some point c in this removed interval such that $a < c < b$; this $c \notin C$. Define $A = (-\infty, c) \cap C$ and $B = (c, \infty) \cap C$. Note, A and B are separated sets, $a \in A, b \in B$, and $A \cup B = C$. Thus, by Definition 3.9, C is totally disconnected.

Alternatively, consider the sum of all intervals, or equivalently, measures, removed by the process of constructing the Cantor set:

$$\sum_{n=0}^{\infty} \left(\frac{2^n}{3^{n+1}}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)\left(\frac{2^n}{3^n}\right) = \left(\frac{1}{3}\right)\left(\frac{1}{1-\frac{2}{3}}\right) = 1$$

Note, the Lebesgue measure of $[0,1]$ is 1, and the measure of the intervals removed is also 1. We are left with a set of measure 0, or a set of only points. Consequently, we have 2 cases left following from the definition of a totally disconnected set. There exists a subspace (singletons) of C that is connected, or the subspaces (singletons) of C are disconnected. If any subspace of C is connected, then C is totally disconnected because there isn't a non-singleton set that is connected. If the subspaces in C are not connected, then C is vacuously totally disconnected as there is no connected subspace of C at all.

Proposition 3.15: *The Cantor set is compact.*

Proof: Recall that a set is compact in $\mathbb{R}$ if and only if it is closed and bounded. C is bounded as $C \subset [0,1]$. C is closed as it is an intersection of closed sets. Therefore, C is compact.

Proposition 3.16: *The Cantor set has no isolated points.*

Proof: It suffices to show that there is no open interval U of $\mathbb{R}$, such that, with respect to the subspace topology on C, $U \cap C = \{x\}$, for some $x \in C$

Recall the definition of a subspace topology. In this case, the topology on C is the intersection of all open sets of $\mathbb{R}$ with C. Let U be some interval $(x - \varepsilon, x + \varepsilon)$ for some $\varepsilon > 0$ and $x \in C$. The point *x* must be some element of *C*, for if it isn't, for some sufficiently small epsilon, $U \cap C = \emptyset$.

To guarantee $U \cap C$ is non-empty, we require U to be centered at some point $x \in C$. I assert that for any such U, $U \cap C$ contains at least 2 points. We divide this proof into 2 cases: *x* is an endpoint, or *x* is not an endpoint.

Case 1: Suppose $x \in C$ is an endpoint of some A_k. Then $x \in U \cap C$; we need to find one more point $y \in U \cap C$. Recall that, in the construction of the Cantor set, the removed intervals have length $\frac{1}{3^n}$ for some $n \in \mathbb{N}$. Choose some n such that $\frac{1}{3^n} < \varepsilon$; then in the n+1st step of constructing C you will find some interval $(c, d) \subset (x - \varepsilon, x + \varepsilon)$ removed in A_n (draw a picture if this is not obvious). From this, we see that *c* or *d* are in the intersection $(x - \varepsilon, x + \varepsilon) \cap C$. Without loss of generality, assume *c* is in the intersection. Then $x, c \in (x - \varepsilon, x + \varepsilon) \cap C$, for any $\varepsilon > 0$. Therefore, in this case, there is no interval $(x - \varepsilon, x + \varepsilon)$ open in $\mathbb{R}$ such that $U \cap C = \{x\}$, for any endpoint $x \in \mathbb{R}$.

Case 2: Suppose $x \in C$ is not an endpoint. What we would like to find is an endpoint in $U \cap C$ for any $\varepsilon > 0$. If *x* is not an endpoint, then for any step *n*, $x \in A_n$. More specifically, there is an interval $[a, b] \subset A_n$ such that $x \in [a, b]$. Choose an $n \in \mathbb{N}$ such that for some interval $x \in [a, b]$ and $[a, b] \subset U$. Then, $a, b, x \in U \cap C$, for any $\varepsilon > 0$.

Because we showed that there is no singleton set open in *C*, *C* consequently has no isolated points.

Proposition 3.17: *The Cantor set is uncountable.*

Proof: *C* is Hausdorff, has no isolated points, and is compact. From Theorem 3.12, *C* is uncountable.

4. Separation Axioms

Definition 4.1: Let X be a topological space. It is called *Hausdorff* or T_2 if for each distinct $x, y \in X$, there exist disjoint sets A and B such that $x \in A \text{ and } y \in B$.

> One example of a Hausdorff space is $\mathbb{R}$. For any $x, y \in \mathbb{R}$, there exist disjoint open sets $\left(x - \frac{|x-y|}{2}, x + \frac{|x-y|}{2}\right), \left(y - \frac{|x-y|}{2}, y + \frac{|x-y|}{2}\right)$ containing x and y, respectively.

Theorem 4.2: A subspace of a Hausdorff space is Hausdorff.

Proof: Let *X* be a Hausdorff space. Let $Y \subset X$ be equipped with the subspace topology $\mathcal{T}_Y$. Let $x, y \in Y$. Then, $x, y \in X$. Because X is Hausdorff, there exist sets $A \text{ and } B$ open in *X* such that $x \in A \text{ and } y \in B, A \cap B = \emptyset$. As $x, y \in Y$, then $A \cap Y, B \cap Y$ are open in *Y*. Note, though, that $(A \cap Y) \cap (B \cap Y) = \emptyset$. To show this, suppose that $(A \cap Y) \cap (B \cap Y) \neq \emptyset$. Then, as $Y \subset X$,

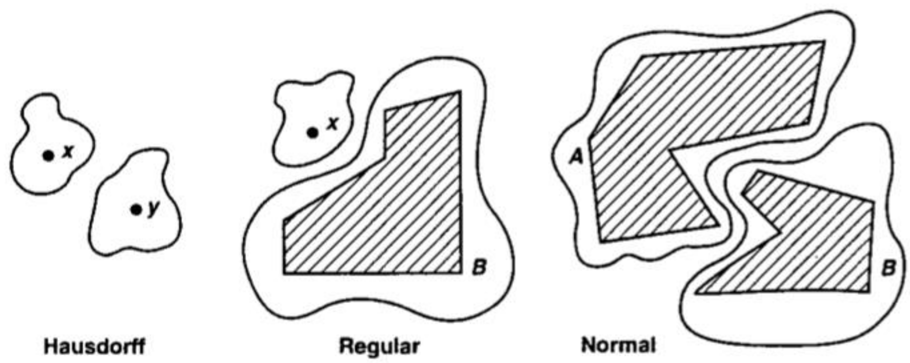
$A \cap B \neq \emptyset$, which contradicts our assumption that X is Hausdorff. So, $(A \cap Y) \cap (B \cap Y) = \emptyset$ and, $x \in A \cap Y, y \in B \cap Y$. Because we chose arbitrary points in Y, we know that Y is Hausdorff.

Definition 4.3: Let X be a topological space and suppose that one-point sets are closed. It is called *regular* or T_3 if for each $x \in X$, and closed set U (such that $x \notin U$), there exist disjoint open sets A and B such that $x \in A \text{ and } U \subset B$.

> $\mathbb{R}$ is also regular (and normal) under the standard topology. A space equipped with the discrete topology is regular (and normal). Every finite regular space is also normal. The Sorgenfrey plane is regular but not normal (the proof for this is challenging).

Definition 4.4: Let X be a topological space and suppose that one-point sets are closed. It is called *normal* or T_4 if for each distinct closed set, U and V, there exist disjoint open sets A and B such that $V \subset A \text{ and } U \subset B$.

Figure 4.5:



Lemma 4.6: Let X be a topological space, and let one-point sets be closed in X

1) X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subset U$
2) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\overline{V} \subset U$

Proposition 4.7: *Let X be a regular space. Then for any distinct $x, y \in X$, x and y have neighborhoods whose closures are disjoint.*

Proof: Let A be some neighborhood of x. Then, because X is regular, there exists some open sets B and C such that $x \in \overline{A} \subset B, y \in C \text{ and } B \cap C = \emptyset$. By Lemma 4.6, there exists some open set D such that $y \in \overline{D} \subset C$. Note that, because $B \cap C = \emptyset$, $\overline{A} \cap \overline{D} = \emptyset$. Then A and D are neighborhoods of x and y such that their closures are disjoint.

Proposition 4.8: *Let X be a normal space. Then for any disjoint closed sets A and B, A and B have neighborhoods whose closures are disjoint.*

Proof: Let A and B be disjoint closed sets. Then, there exists open sets U and V such that $A \subset U, B \subset V, U \cap V = \emptyset$. By Lemma 4.6, there also exist open sets W, Z such that $A \subset \overline{W} \subset U$, and $B \subset \overline{Z} \subset V$. Then, as $U \cap V = \emptyset$, $\overline{W} \cap \overline{Z} = \emptyset$. Therefore, for any disjoint closed set A and B, there exist open neighborhoods W and Z such that $\overline{W} \cap \overline{Z} = \emptyset$.

Theorem 4.9: *A closed subspace of a normal space is normal.*

Proof: Let Y be a closed subspace of a normal space X. Take some disjoint closed sets $A, B \subset Y$; then $A = A_X \cap Y, B = B_X \cap Y$. But A_X and B_X must be closed in X. Then, there exist sets C_X, D_X open in X such that $A_X \subset C_X, B_X \subset D_X, and\ C_X \cap D_X = \emptyset$. Then define sets $C = C_X \cap Y$ and $D = D_X \cap Y$. C and D are open in Y and are disjoint open sets that separate A and B. Because we showed this for arbitrary closed sets A and B, we may conclude that the closed subspace of a normal space is normal.

Lemma 4.10: *A subspace of a regular space is regular*

Proof: Let X be a regular space and let $Y \subset X$. Choose some point $x \in Y$ and closed set $U \subset Y$; then $x \in X$, and $\mathrm{U} = \mathrm{V} \cap X$, for some set V closed in X. Then, as X is regular, there exist disjoint open sets A, B such that $x \in A\ and\ V \subset B$. Then, $A \cap Y, B \cap Y$ are disjoint open sets that contain x and U, respectively.

Theorem 4.11: *X and Y are regular if and only if $X \times Y$ is regular.*

Proof: Suppose X and Y are regular. Choose some point $x \times y \in X \times Y$ and a neighborhood U of $x \times y$. U may be written as a product of open sets $A \times B$, where A is open in X, and B is open in Y. Note, $x \in A$ and $y \in B$. By Lemma 4.6, there exist open sets C, D such that $x \in \overline{C} \subset A$ and $y \in \overline{D} \subset B$. Then, it is easy to see that $x \times y \in \overline{C} \times \overline{D} \subset A \times B$, which by Lemma 4.6 implies that $X \times Y$ is regular, as $\overline{C} \times \overline{D} = \overline{C \times D}$.

For the other direction, suppose $X \times Y$ is regular. Then $X \times \{y\}$ is regular by Lemma 4.10; but it can be readily shown that $X \times \{y\}$ is homeomorphic to X, so X is regular. Again, take a subspace $Y \times \{x\}$, this is regular and homeomorphic to Y, so Y is regular.

In fact, it can be proven that the arbitrary product of regular spaces is regular, not just the finite product.

5. References

All figures, exercises, theorems, and definitions come directly from *Topology* by Munkres.

- Munkres, James Raymond. *Topology*. Pearson, 2018.